Bosonic string with antisymmetric fields and a non-local Casimir effect

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Abstract

The coupling, in a non-standard way, of a bosonic string theory with a dilaton and antisymmetric fields is investigated. By integrating over the antisymmetric fields, a Coulomb-like interaction term is generated. The static potential of a theory of this kind is obtained from the corresponding non-local zeta function, in some approximation. An interpretation of the static potential as a type of non-local Casimir effect is given.

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1. Introduction. It escapes to nobody nowadays that, in spite of its long history, string theory (see [1] for a general review) is not a completely understood discipline yet. This is reflected, in particular, by the fact that different versions, formulations and modifications of string theories are still being actively studied. Among the most interesting, recent modifications to this theory one should include the cases of a string coupled to a background field [2], the rigid string [3], and the Dirichlet string theory [4]. Some months ago, the coupling of a rigid string to antisymmetric fields was discussed, Ref. [5]. This formulation is certainly different from the by now standard case of a string in a background field [2], because in Ref. [5] the kinetic term for the antisymmetric fields had to be added by hand. Later, the antisymmetric fields can be integrated out, what leads to a Coulomb-like term in the potential.

In the present paper we shall consider the usual bosonic string theory, interacting, in the above way, both with antisymmetric and dilaton fields. The precise form of the interaction of Coulomb type that appears after integrating out the background fields will be found. We shall see that the static potential in such a theory leads to a very non-standard zeta function, which can be interpreted as originating a non-local Casimir effect. A careful (although necessarily approximate) study of this static potential —which arises from the corresponding (non-local) zeta function— will be given.

2. String theory coupled to a dilaton and antisymmetric fields. Consider the action of a closed bosonic string in a field of massless modes:

$$S = k \int d^2\xi \, \sqrt{g} \, \left[\frac{1}{2} G_{ij}(X) g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j + \zeta R^{(2)} \phi(X) + \frac{e}{\sqrt{g}} \epsilon^{\mu\nu} A_{ij} \partial_\mu X^i \partial_\nu X^j \right], \qquad (1)$$

where $i = 1, \ldots, D$ and $\mu = 1, 2$, G_{ij} is a symmetric tensor (the source of the graviton modes), k is the string tension (which we will set equal to 1 in this section), A_{ij} is an antisymmetric tensor (the source for the antisymmetric 2-tensor modes), and $\phi(x)$ is the dilaton field (notice that we do not consider the tachyon). In the standard approach to the string effective action [2, 1], the kinetic terms for the sources that appear in (1) would show up explicitly after integrating over the metrics.

However, our approach here will be different. Let us imagine that string theory is coupled to some fields —which can simply be considered as external ones. In particular, we may consider a theory coupled to a dilaton and Kalb-Ramond fields [6], in which case one has to

add to the action (1) the corresponding kinetic terms for those fields:

$$S_{kin} = \int d^D y \sqrt{g(y)} \left[\frac{1}{2}\phi \Box_D \phi + \frac{1}{12} F_{ijk} F^{ijk} \right], \tag{2}$$

where F_{ijk} is the stress tensor for A_{ij} and where the kinetic terms in (2) are defined in a D-dimensional curved spacetime, being $\Box_D$ the d'Alembertian in such spacetime. As usually, G_{ij} is a more fundamental object than A_{ij} and ϕ, since it plays the role of a kinetic term for the bosonic string. This is the reason why we do not need to add any term of the form (2) for it. Furthermore, there is no possibility of guessing a simple kinetic term corrsponding to G_{ij}, due to its geometrical structure.

The source terms in (1) can be written as follows

$$\int d^D y \sqrt{g(y)} \left[K(y)\phi + \frac{1}{\sqrt{g}} K^{ij}(y) A_{ij} \right], \tag{3}$$

where

$$\begin{aligned} K(y) &= \int d^2\xi \sqrt{g} \left[\zeta R^{(2)}(\xi) \delta(y - X(\xi)) \right], \\ K^{ij}(y) &= \int d^2\xi \left[e\, \epsilon^{\mu\nu} \partial_\mu y^i \partial_\nu y^j \delta(y - X(\xi)) \right]. \end{aligned} \tag{4}$$

One can see that the functional integrals over A_{ij} and ϕ have the standard Gaussian form. Thus, we can integrate over A_{ij} and ϕ in order to obtain an effective theory for the closed bosonic string, exhibiting a Coulomb-like interaction term (compare with Ref. [5] and we also suppose that $D = 4$,otherwise the form of potential will be different)

$$\begin{aligned} S &= \int d^2\xi \sqrt{g} \left[\frac{1}{2} G_{ij}(X) g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j \right] + S_{int}, \\ S_{int} &= \int d^2\xi d^2\xi' \left[c_1 e^2 \sigma_{ij}(\xi) \sigma^{ij}(\xi') + c_2 \zeta^2 {R^{(2)}}^2 \right] V(|x - x'|), \\ &\sigma_{ij}(\xi) = \epsilon^{\mu\nu} \partial_\mu X^i \partial_\nu X^j, \qquad V(|x - x'|) = \frac{1}{|x(\xi) - x(\xi')|^2 + a^2}, \end{aligned} \tag{5}$$

being c_1 and c_2 non-essential numerical constants which can be choosen to be equal to 1. It is the coupling of the bosonic string with the kinetic terms of the form (2) what induces the higher derivatives in the effective action and the non-local interaction term. In what follows we shall consider, for simplicity, the situation where the two-dimensional space is flat, and hence $R^{(2)} = 0$. Notice that in the potential, V, a cut-off parameter, a^2, has been introduced, in order to avoid the singularity that occurs at $x = x'$. That is just usual ultraviolet-type cut-off. The appearance of a term of the form of the one obtained here —induced by the

antisymmetric tensor field— has been mentioned in Ref. [5] in connection with the rigid string theory.

An interesting question is now, to which consequences can this Coulomb-like term lead in the frame of string theory. We are going to discuss this question in some detail, by means of the evaluation of the static potential for the model (5) in the case of a flat metric $G_{ij} = \delta_{ij}$.

3. The static potential, from a non-local zeta function. The static potential in string theory is an interesting magnitude in connection with the possible applications of string theory to QCD. This fact was realized long ago [7], and a calculation of the static potential in different string models has been carried out explicitly in Refs. [8, 9], and in particular for the rigid string [3] in Refs. [10]-[12]. It has been pointed out in those works that the leading corrections to the static potential have a universal character [8].

In the applications of the static potential to QCD one can usually choose the Wilson loop C to be a rectangle on the plane, of length T and width R (with $T >> R$). Then, the loop expectation value can be found as follows [13]:

$$W[C] \sim \exp[-TV_s(R)]. \tag{6}$$

The explicit one-loop calculation for a standard bosonic string gives:

$$V_s(R) = kR + \frac{D-2}{2} \mathrm{Tr}\ \ln \Box, \tag{7}$$

where k is again the string tension and $\Box$ the two-dimensional d'Alembertian in the space of topology $R \times S^1$. Using the zeta-function regularization procedure to calculate (7) one obtains the well-known result [8, 9, 14]

$$V_s(R) = kR - \frac{(D-2)\pi}{24R}. \tag{8}$$

We shall now perform the calculation to one-loop of the static potential (7) taking into account the Coulomb-like term induced by the antisymmetric tensor fields, as in (5). We discover that the integration over the X^is is not Gaussian any more and this makes the calculation highly non-trivial. We shall however prove that a meaningful result can be obtained in a quite reasonable approximation. The natural way to consider the integration is by decomposing the variables as follows:

$$X^i(\xi) = X_0^i(\xi) + X_1^i(\xi), \tag{9}$$

where $X_0^i(\xi)$ is a background variable, which is a linear function of ξ satisfying the field equations

$$X_0^i(\xi) = c_\mu^i \xi^\mu, \qquad \eta_{ij} c_\mu^i c_\nu^j = \eta_{\mu\nu}, \tag{10}$$

being the c_μ^i some constants. The expansion of Eq. (5) up to second order on the fluctuations $X_1^i(\xi)$ can be performed in the same way as it was done in Ref. [5]. After the subsequent functional Gaussian integration over the X_1^is, we obtain the static potential (for simplicity we consider below only case $D = 4$)

$$V_s(R) = kR + \frac{D-2}{2} \int dp \sum_{n=1}^{\infty} \ln \left[\frac{p^2}{2} + 2e^2 p^2 \int d^2\xi \, \frac{e^{ip\cdot\xi}}{\xi\cdot\xi + a^2} + 4e^2 \int d^2\xi \, \frac{e^{ip\cdot\xi} - 1}{(\xi\cdot\xi + a^2)^2} \right]. \tag{11}$$

Notice that the notation has been simplified somehow, because in (11) it must be understood that the double integration over $d^2\xi$ is in fact a single integration on the first coordinate ξ_1 and an infinite sum (one of the spatial coordinates corresponds to the torus), exactly as in the case of the first integration (over p and n). For the benefit of the reader, let us here briefly illustrate this case, where the procedure of zeta-function regularization [15, 16] is self-explanatory (for a very detailed review, see [17])

$$\begin{aligned} \int d^2p \; (p\cdot p)^{-s} &= \int_0^\infty dp \sum_{n=1}^{\infty} \left(p^2 + \frac{n^2}{R^2} \right)^{-s} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} e^{-n^2 t/R^2} \int_0^\infty \sum_{n=1}^{\infty} dp \; e^{-p^2 t} = -\frac{\sqrt{\pi}\,\Gamma(s-1/2) R^{2s-1}}{2\,\Gamma(s)} \, \zeta(2s-1), \end{aligned} \tag{12}$$

and taking the derivative at $s = 0$, this yields the result

$$\int d^2p \, \ln(p\cdot p) = \int_0^\infty dp \sum_{n=1}^{\infty} \ln \left(p^2 + \frac{n^2}{R^2} \right) = -\frac{\pi}{12R}, \tag{13}$$

a particular case of (8). The double integral $d^2\xi$ and products $p\cdot\xi$ and $\xi\cdot\xi$, in the much more complicated expression (11), are to be dealt with in a similar way. Let us introduce the basic integrals

$$\begin{aligned} I_1 &\equiv \int d^2\xi \frac{e^{ip\cdot\xi}}{\xi^2 + a^2} = \int_0^\infty dq \sum_{m=1}^{\infty} \frac{e^{ipq + nm/R}}{q^2 + m^2 + a^2} = \frac{\pi}{2} \sum_{m=1}^{\infty} \frac{e^{inm/R} e^{-p\sqrt{m^2+a^2}}}{\sqrt{m^2 + a^2}}, \\ I_2 &\equiv \int d^2\xi \frac{e^{ip\cdot\xi} - 1}{(\xi^2 + a^2)^2} = \int_0^\infty dq \sum_{m=1}^{\infty} \frac{e^{ipq + nm/R} - 1}{(q^2 + m^2 + a^2)^2} \\ &= \frac{\pi}{4} \sum_{m=1}^{\infty} \left[\frac{e^{inm/R} e^{-p\sqrt{m^2+a^2}} \left(1 + p\sqrt{m^2 + a^2}\right)}{(m^2 + a^2)^{3/2}} - \frac{1}{2(m^2 + a^2)^{3/2}} \right]. \end{aligned} \tag{14}$$

By plugging them back into Eq. (11), we obtain

$$
\begin{aligned}
V_s(R) &= kR - \frac{(D-2)\pi}{24R} + \frac{D-2}{2}\int dp \sum_{n=1}^{\infty} \ln \left\{ 4\pi e^2 \sum_{m=1}^{\infty} \frac{e^{inm/R} e^{-p\sqrt{m^2+a^2}}}{\sqrt{m^2+a^2}} \right. \qquad (15) \\
&\left. + \frac{4\pi e^2}{p^2+n^2/R^2} \sum_{m=1}^{\infty} \left[\frac{e^{inm/R} e^{-p\sqrt{m^2+a^2}} \left(1+p\sqrt{m^2+a^2}\right) - 1}{(m^2+a^2)^{3/2}} - \frac{1}{2(m^2+a^2)^{3/2}} \right] \right\}.
\end{aligned}
$$

No approximation is involved in Eq. (15). However, this expression is quite complicated and to proceed further we need to do some approximation, valid in the limit when a is big ($a > R$). In this case the sums can be substituted by integrals, which can then be calculated exactly (although the results are rather long and uninteresting and will not be written here). It is more sensible to discuss the first approximation in $1/a$, obtained by keeping just the leading terms of the final result. This yields

$$
V_s(R) \simeq kR - \frac{(D-2)\pi}{24R} \left[1 - \frac{6\pi e^2 R^2}{a^3} \left(\gamma + 2 \text{ Sinint}\,(1) - \pi + e^{-a/R} \right) - \frac{24\pi e^2}{a(e^{a/R}-1)} \right], \quad (16)
$$

where γ is Euler's constant and Sinint the standard sinus integral (Sinint $(1) = 0.946083$). (Observe that in this expression we have kept a couple of terms which are representative of the asymptotically smaller contributions to the effective potential that can be dismissed completely in this approximation.) The dependence on a and R could have been ascertained by dimensional reasons. Numerically, the result is:

$$
V_s(R) = kR - \frac{(D-2)\pi}{24R} \left\{ 1 - \frac{e^2 R^2}{a^3} \left[25.3416 + \mathcal{O}\left(\frac{1}{a}\right) \right] \right\}. \quad (17)
$$

Having at hand this result for the effective potential, we can now study in some detail the contribution of the antisymmetric fields to the static potential. From (17) we see immediately that the static potential is given by expression (8) with a renormalized string tension, namely

$$
V_s(R) = k_R R - \frac{(D-2)\pi}{24R}, \quad (18)
$$

where

$$
k_R \simeq k + 3.3172\,(D-2)\,\frac{e^2}{a^3}. \quad (19)
$$

Hence, we observe that when the radius R equals R_c, $R_c^2 = (D-2)\pi/(24k_R)$, it turns out that $V_s(R_c) = 0$. The appearance of this critical radius, R_c indicates very probably —as in more complicated string models— that the quasi-static string picture ceases to be valid there, what has been interpreted in Ref. [9] (using a different string model as example) as

a signal for a phase transition. From this point of view, it is now natural to interprete the effect of the antisymmetric fields in the static potential as a renormalization of the string constant k (a one-loop correction to the classical potential), what produces a change in the value of the critical radius R_c, as compared with the one that it has in the case when there is no coupling with antisymmetric fields. The fact that we have been able to obtain in a quite precise way the magnitude of this modification is also to be remarked. To our knowledge, this is the first time that such kind of non-local calculation has been performed by using standard zeta-function methods. And it is well known that subleading effects, as those that we get here, can be certainly relevant for the study of phase transitions, because they may change in some cases the nature of the phase transition itself.

4. Conclusion. This finishes our preliminar investigation of a string theory coupled to dilatonic and antisymmetric fields where, after integration over these last ones, an effective term in the potential, of Coulomb form, is generated. We have discussed the influence of this term in the static potential and we have calculated it by using the zeta-function regularization procedure. The result can be interpreted as corresponding to a new kind of non-local Casimir effect. Indeed, let us consider a two-dimensional scalar field with $(D-2)$-components, φ^i, defined in the space $R \times S^1$, with the following non-local Lagrangian:

$$L = \varphi_i(x) \left[-\Box_x - 4e^2 \Box_x \int d^2y \, \frac{e^{-y^\mu \partial_\mu^x}}{y^2 + a^2} + 8e^2 \int d^2y \, \frac{e^{-iy^\mu \partial_\mu^x} - 1}{(y^2 + a^2)^2} \right] \varphi^i(x), \tag{20}$$

where $i = 1, \ldots, D-2$. The non-local Casimir effect in such a theory is obtained precisely through the non-local zeta-function corresponding to (11), in the same way as the previous calculation that has been carried out in Sect. 3. It also has the usual meaning as a quantum correction to the free energy.

Summing up, we conclude that this study of a non-local Lagrangian —in connection with the zeta function regularization method— opens new possibilities to extend the fundamental concepts of vacuum energy or Casimir effect to completely new configurations in non-local settings. From a different point of view, our results show clearly that when a string is coupled to other fields, the corresponding effective field theory, which approximately describes such a picture, might easily be a non-local one. Finally, the remarkable power of the zeta-function techniques is very useful in order to deal with such —otherwise intractable— situations, opening a promising new path for further developements.

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